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On the collective instability of salt fingers

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In this paper we consider the stability of salt fingers to long wavelength internal wave perturbations. The Prandtl number of the fluid is assumed to be large, but the ratio of the two diffusivities (K_S/K_T) is allowed to be any size provided that $K_S < K_T$. This problem was first considered by Stern (1969), where several untested assumptions were made about the motion. Here we use a two-scale approach to separate the salt finger motions from the long-scale internal wave perturbations and to obtain the stability criterion. This collective instability of salt fingers succeeds in transferring energy from the small salt finger scales to the long internal wave scales.

1. Introduction

The salt finger mechanism was first discussed by Stommel, Arons & Blanchard (1956), when it was considered to be an oceanographic curiosity of little practical or scientific importance. The first theoretical discussion of salt fingers was given by Stern (1960). Since this time, the subject of thermohaline convection has been studied both theoretically and experimentally by a number of authors, and it is now recognized as a feature of major importance in transport processes in the ocean. Salt fingers have been observed in the Mediterranean outflow by Williams (1974), where thin salt-finger regions, about 20 cm thick, are separated by convecting regions several metres thick.

Salt fingers can be formed when a layer of hot, salty fluid lies above a layer of cold, fresh fluid of greater density. If a parcel of fluid in this system is displaced upwards, it will come into thermal equilibrium with its surroundings before it comes into saline equilibrium, because the diffusivity of heat is greater than that of salt. It will then be fresher than its surroundings, but at the same temperature, so it will continue to rise. By an analogous argument, if a parcel of fluid is displaced downwards, it will continue to fall. Salt fingers are the long, thin cells of alternating upward and downward motion which occur by the above mechanism.

The problem we wish to consider here is the stability of these salt fingers to long wavelength internal wave perturbations. Stern (1969, 1975) made an *ad hoc* investigation of this instability, which he named the collective instability of salt fingers. In his study, Stern made a series of untested assumptions about the coupling between the salt fingers and the large-scale motions. In the work presented here, a complete study of the problem is made, using an averaging procedure to link the large and the small scales. We find that there is instability if

$$\frac{\beta F_S - \alpha F_T}{\nu(\alpha T_z - \beta S_z)} > \frac{1}{3}, \tag{1.1}$$

where F_T and F_S are the heat and salt fluxes of the salt fingers, ν is the kinematic viscosity of the fluid, and T_z and S_z are the heat and salt gradients in the fluid. This result is slightly different from that of Stern, who obtained the instability criterion (Stern 1975, equation (11.3.13))

$$\frac{\beta F_S - \alpha F_T}{\nu(\alpha T_z - \beta S_z)} > 1. \tag{1.2}$$

He made the assumptions that the salt fingers rotate with the internal wave, and that the magnitude of the flux associated with them remains unaltered. We show that the heat and salt fluxes are, in fact, increased by the internal wave. The model used here is idealized to the extent that only two-dimensional motions of the salt fingers and the internal wave are considered. The Prandtl number, ν/K_T, the ratio of viscosity to thermal diffusivity, is assumed to be large.

2. The salt-finger solution

Suppose we have an unbounded region of incompressible fluid which has a stable linear temperature gradient, T_z, and an unstable linear salt gradient, S_z, with the overall density statically stable, i.e. increasing with depth. The co-ordinate vertically upwards shall be taken as z and the horizontal co-ordinate shall be x. We shall consider only two-dimensional motions, so we can define a stream function, ψ, by

$$u = -\frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial x}, \tag{2.1}$$

where u is the horizontal velocity and w is the vertical velocity in the fluid. The temperature field, T', and the salinity field, S', will be given by

$$T' = T_z z + T(x,z,t), \quad S' = S_z z + S(x,z,t). \tag{2.2}$$

The density field will be given by

$$\rho = \rho_0(1 - (\alpha T_z - \beta S_z) z - (\alpha T - \beta S)), \tag{2.3}$$

where α and β are the coefficients of expansion for heat and salt, defined so α and β are both positive. The two-dimensional equations of motion are then

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) = \frac{\partial}{\partial x}(g(\alpha T - \beta S)) + \nu\nabla^4\psi, \\ &\frac{\partial T}{\partial t} + J(\psi, T) + T_z\frac{\partial \psi}{\partial x} = K_T\nabla^2 T, \\ &\frac{\partial S}{\partial t} + J(\psi, S) + S_z\frac{\partial \psi}{\partial x} = K_S\nabla^2 S, \end{aligned}\right\} \tag{2.4}$$

where

$$J(\psi, \phi) = \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial x},$$

the Jacobian, and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$. The thermal diffusivity is K_T and the saline diffusivity is K_S.

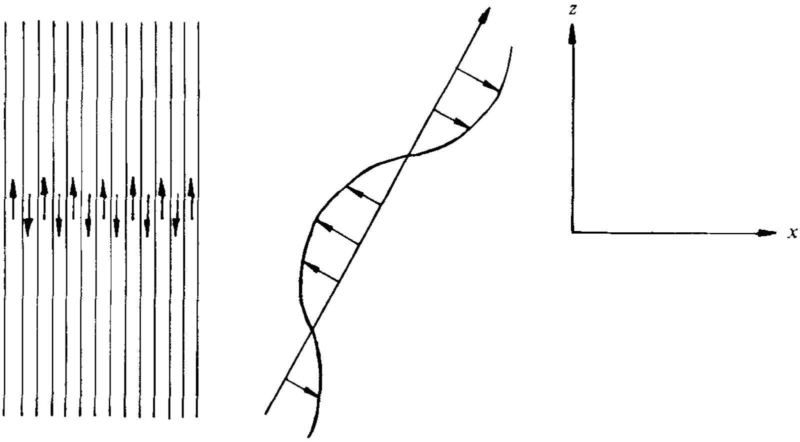

FIGURE 1. (a) The basic salt-finger state. The temperature field is given by $T' = T_z z + \hat{T} \sin x$ and the salt giled by $S' = S_z z + \hat{S} \sin x$. The motions are purely vertical, and the vertical velocity is given by $w = \hat{W} \sin x$. (b) The long internal wave perturbation interacts with the fingers eventually causing the growth of the internal wave.

We now non-dimensionalize the equations (2.4). We choose a length scale l, which will be the horizontal length scale of the salt fingers, and a time scale l^2/K_T. Then the equations (2.4) become

$$\left.\begin{aligned} \frac{1}{\sigma}\left[\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi)\right] &= R_T\frac{\partial T}{\partial x} - R_S\frac{\partial S}{\partial x} + \nabla^4\psi, \\ \frac{\partial T}{\partial t} + J(\psi, T) + \frac{\partial \psi}{\partial x} &= \nabla^2 T, \\ \frac{\partial S}{\partial t} + J(\psi, S) + \frac{\partial \psi}{\partial x} &= \tau\nabla^2 S, \end{aligned}\right\} \tag{2.5}$$

where all the quantities appearing in (2.5) are now dimensionless and

$$\left.\begin{aligned} \sigma &= \frac{\nu}{K_T}, & \tau &= \frac{K_S}{K_T}, \\ R_T &= \frac{\alpha g T_z l^4}{\nu K_T}, & R_S &= \frac{\beta g S_z l^4}{\nu K_T}. \end{aligned}\right\} \tag{2.6}$$

We look for a steady solution to the equations (2.5) which represents the motion in the salt fingers, and is a function of x only. We try

$$\psi = -\hat{W}\cos x, \quad T = \hat{T}\sin x, \quad S = \hat{S}\sin x, \tag{2.7}$$

where $\hat{W}$, $\hat{T}$ and $\hat{S}$ are constants. Substituting (2.7) into (2.5), we find (2.7) is a solution provided

$$\hat{W} = R_T\hat{T} - R_S\hat{S} \tag{2.8a}$$

and

$$\hat{T} = -\hat{W}, \quad \hat{S} = -\frac{\hat{W}}{\tau}. \tag{2.8b}$$

These imply

$$\frac{R_S}{\tau} - R_T = 1. \tag{2.9}$$

The equations (2.7) with the relationships (2.8)–(2.9) give a steady solution to the fully nonlinear equations (2.5). Huppert & Manins (1973) have shown that under certain conditions, this solution is the only solution to the two-dimensional equations of motion. The solution is sketched in figure 1.

The vertical heat flux in the fingers is given by

$$F_T = -\overline{\frac{\partial \psi}{\partial x} T}\,\frac{K_T}{l}\, lT_z,$$

where $\overline{(\ )}$ denotes a horizontal average over many wavelengths of the salt fingers. Then

$$F_T = -K_T T_z \hat{W}\hat{T}\,\overline{\sin^2 x} = -\tfrac{1}{2}\hat{W}\hat{T}K_T T_z. \tag{2.10}$$

Similarly, the vertical salt flux

$$F_S = -\overline{\frac{\partial \psi}{\partial x} S}\, K_T S_z = -\tfrac{1}{2}\hat{W}\hat{S}K_T S_z. \tag{2.11}$$

By (2.8*b*) we then see

$$\frac{\beta F_S}{\alpha F_T} = \frac{\beta \hat{S} S_z}{\alpha \hat{T} T_z} = \frac{1}{\tau}\frac{R_S}{R_T}. \tag{2.12}$$

Now the ratio $\beta F_S/\alpha F_T$ is also the ratio of the potential energy lost by the salt field to the potential energy gained by the temperature field. By equation (2.9)

$$\frac{R_S}{\tau} > R_T.$$

Hence

$$\frac{\beta F_S}{\alpha F_T} > 1,$$

and so we see that the potential energy lost by the salt field is greater than the potential energy gained by the temperature field. This must be the case, since otherwise the system would be gaining energy.

3. The averaging procedure

We now perturb the salt-finger solution by putting

$$\left.\begin{aligned} \psi &= -\hat{W}\cos x + \psi(x,z,t),\\ T &= \hat{T}\sin x + T(x,z,t),\\ S &= \hat{S}\sin x + S(x,z,t). \end{aligned}\right\} \tag{3.1}$$

Substituting (3.1) into (2.5) and linearizing in the perturbation gives

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2\psi - R_T\frac{\partial T}{\partial x} + R_S\frac{\partial S}{\partial x} = -\frac{\hat{W}}{\sigma}\sin x\left(\frac{\partial}{\partial z}\nabla^2 + \frac{\partial}{\partial z}\right)\psi, \tag{3.2a}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)T + \frac{\partial \psi}{\partial x} = -\hat{W}\sin x\frac{\partial T}{\partial z} - \hat{W}\cos x\frac{\partial \psi}{\partial z}, \tag{3.2b}$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)S + \frac{\partial \psi}{\partial x} = -\hat{W}\sin x\frac{\partial S}{\partial z} - \frac{\hat{W}}{\tau}\cos x\frac{\partial \psi}{\partial z}. \tag{3.2c}$$

We now write the equation (3.2) in the operator form

$$Lu = Mu \tag{3.3}$$

where L is the linear operator on the left-hand side of (3.2), M is the linear 'rapidly varying' operator on the right-hand side of (3.2), which is due to the salt-finger field, and $u = (\psi, T, S)^T$.

Since the coefficients in (3.3) are independent of z and t, we can find solutions with

$$u = \mathscr{R}(u(x)\, e^{imz+i\omega t}),$$

where m is a vertical wavenumber and ω is the wave frequency. We wish to consider perturbations, u, which vary over a horizontal length scale much larger than l, the width of the salt fingers. However, the salt-finger field forces motions which vary on the short length scale, l. So we put

$$u(x) = u_m(x) + u_r(x), \tag{3.4}$$

where $u_m(x)$ is the mean part of the field, which varies over a horizontal length scale, $1/k$, and $u_r(x)$ is the rapidly varying part, which varies on a length scale, l. For the mean field we shall look for wave solutions

$$u_m = \begin{pmatrix} \psi_m \\ T_m \\ S_m \end{pmatrix} = \mathscr{R}\left(\begin{pmatrix} A \\ -iB \\ -iC \end{pmatrix} \exp(ikx + imz + i\omega t)\right). \tag{3.5}$$

The wavenumber of this wave is given by μ, where $\mu^2 = k^2 + m^2$. It is travelling at an angle θ to the vertical where $k = \mu \sin\theta$ and $m = \mu\cos\theta$. The basis of our approximations will be $\mu \ll 1$.

We define a new co-ordinate system (x', z') with

$$\left.\begin{aligned} x' &= x\sin\theta + z\cos\theta, \\ z' &= -x\cos\theta + z\sin\theta. \end{aligned}\right\} \tag{3.6}$$

In this system, x' measures distance in the direction of wavenumber vector and z' measures distance perpendicular to this direction, along the fronts. Then

$$u_m = \mathscr{R}\left(\begin{pmatrix} A \\ -iB \\ -iC \end{pmatrix} \exp(i\mu x' + i\omega t)\right). \tag{3.7}$$

We define an averaging operator $\langle\ \rangle$ by

$$\langle \rho \rangle = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \rho\, dz'. \tag{3.8}$$

So, from the definition, $\langle u_m \rangle = u_m$ and $\langle u_r \rangle = 0$. Since L is a linear operator with constant coefficients $\langle L\rho \rangle = L\langle \rho \rangle$, so $\langle Lu_m \rangle = Lu_m$ and $\langle Lu_r \rangle = 0$. Also $\langle Mu_m \rangle = 0$ since the rapidly varying operator acting on a mean quantity will give a rapidly varying result. Thus averaging (3.3), we obtain

$$Lu_m = \langle Mu_r \rangle. \tag{3.9}$$

Subtracting (3.9) from (3.3) gives

$$Lu_r = Mu_m + Mu_r - \langle Mu_r \rangle. \tag{3.10}$$

Now we make the approximation $\mu \ll 1$. We then expect $Mu_r - \langle Mu_r \rangle$ to be negligible compared with Mu_m, since u_r is associated with the length scale l and u_m is associated with the length scale $1/\mu$. If this is the case, equation (3.10) becomes

$$Lu_r = Mu_m. \tag{3.11}$$

We shall make this approximation and then in § 4, when we have calculated u_m and u_r explicitly, we verify its validity. The approximation made to obtain (3.11) is equivalent to the first-order smoothing of kinematic magnetohydrodynamics (Moffatt 1978).

In order to solve (3.9) and (3.11), we use the relationship that

$$\mathscr{R}(a)\,\mathscr{R}(b) = \tfrac{1}{2}(\mathscr{R}(ab) + \mathscr{R}(ab^*)) \tag{3.12}$$

where * denotes complex conjugate. Then, from (3.11), we find

$$u_r = \mathscr{R}\left(\begin{pmatrix} iA_1 \\ B_1 \\ C_1 \end{pmatrix} \exp\left(i(k-1)\,x + imz + i\omega t\right) + \begin{pmatrix} -iD_1 \\ E_1 \\ F_1 \end{pmatrix} \exp\left(i(k+1)\,x + imz + i\omega t\right)\right). \tag{3.13}$$

Substituting (3.13) and (3.5) into (3.11), using (2.8*b*) and (3.12), and equating coefficients, we find

$$\left.\begin{aligned} -\left(\frac{i\omega}{\sigma} + \mu_-^2\right)\mu_-^2 A_1 + (1-k)\,(R_T B_1 - R_S C_1) &= -\frac{im\hat{W}}{2\sigma} A, \\ (i\omega + \mu_-^2)\,B_1 + (1-k)\,A_1 &= -\frac{im\hat{W}}{2}(B+A), \\ (i\omega + \tau\mu_-^2)\,C_1 + (1-k)A_1 &= -\frac{im\hat{W}}{2}\left(C + \frac{A}{\tau}\right), \end{aligned}\right\} \tag{3.14}$$

$$\left.\begin{aligned} -\left(\frac{i\omega}{\sigma} + \mu_+^2\right)\mu_+^2 D_1 + (1+k)\,(R_T E_1 - R_S F_1) &= -\frac{im\hat{W}}{2\sigma} A, \\ (i\omega + \mu_+^2)\,E_1 + (1+k)\,D_1 &= \frac{im\hat{W}}{2}(B-A), \\ (i\omega + \tau\mu_+^2)\,F_1 + (1+k)\,D_1 &= \frac{im\hat{W}}{2}\left(C - \frac{A}{\tau}\right), \end{aligned}\right\} \tag{3.15}$$

where $\mu_\pm^2 = 1 \pm 2k$. By substituting (3.13) and (3.5) into (3.9), we obtain

$$\left.\begin{aligned} -\left(\frac{i\omega}{\sigma} + \mu^2\right)\mu^2 A - k(R_T B - R_S C) &= \frac{im\hat{W}}{2\sigma}\left((\mu_-^2 - 1)\,A_1 + (\mu_+^2 - 1)\,D_1\right), \\ (i\omega + \mu^2)\,B - Ak &= \frac{im\hat{W}}{2}\left(-(B_1 - E_1) - (D_1 - A_1)\right), \\ (i\omega + \tau\mu^2)\,C - Ak &= \frac{im\hat{W}}{2}\left(-(C_1 - F_1) - \frac{1}{\tau}(D_1 - A_1)\right). \end{aligned}\right\} \tag{3.16}$$

To obtain these equations we have made no assumptions about the sizes of σ, τ, or ω.

Now (3.14)–(3.16) constitute a set of nine linear homogeneous, simultaneous equations in A, B, C, A_1, B_1, C_1, D_1, E_1 and F_1. Thus, in order for a solution to exist, the determinant of the coefficients of this set of equations must be zero. This determinant will give the dispersion relation $\omega(k,m) = 0$. We can, in fact, somewhat simplify this procedure by solving (3.14) and (3.15) to give A_1, B_1, C_1, D_1, E_1 and F_1 in terms of A, B and C and then using these to substitute into (3.16). We do this in the next section.

We note here that it is possible to do the averaging using a different averaging operator and to obtain the same dispersion relation. If we define $\langle\ \rangle'$ as the horizontal x-average over distances $\mathscr{L}$ such that $l \ll \mathscr{L} \ll 1/k$, i.e.

$$\langle\rho\rangle' = \frac{1}{2\mathscr{L}}\int_{-\mathscr{L}}^{\mathscr{L}}\rho\,dx,$$

provided $\mu \ll 1$, we obtain the equations

$$Lu_m = \langle Mu_r\rangle'$$

and

$$Lu_r = Mu_m$$

and the dispersion relation remains identical.

4. The stability criterion

We now solve (3.14) and (3.15) for $A_1 - F_1$ in terms of A, B and C. We assume now that $\sigma \gg 1$ and $\tau \leqslant O(1)$. Terms containing σ still appear in the equations because we have yet to determine the size of ω in relation to that of σ. We now have

$$\left.\begin{aligned}
(D_1 - A_1)P &= im\hat{W}(BR_T(i\omega+\tau+O(k)) - CR_S(i\omega+1+O(k)) - kSA),\\
(D_1 + A_1)P &= im\hat{W}RA + O(k)\,B + O(k)\,C,\\
(B_1 - E_1)P &= -im\hat{W}CR_S - im\hat{W}B\left(-\frac{\omega^2}{\sigma} + i\omega - \tau R_T\right)\\
&\qquad + \frac{im\hat{W}Ak}{(i\omega+1)^2}(-2P + (i\omega-1)R - (i\omega+1)S),\\
(C_1 - F_1)P &= im\hat{W}BR_T - im\hat{W}C\left(-\frac{\omega^2}{\sigma} + i\omega + \frac{R_S}{\tau}\right)\\
&\qquad + \frac{im\hat{W}Ak}{(i\omega+\tau)^2}(-2P + (i\omega-\tau)R - (i\omega+\tau)S),
\end{aligned}\right\} \tag{4.1}$$

where

$$P = -\frac{i\omega^3}{\sigma} - \omega^2 + i\omega\left(\frac{R_S}{\tau} - \tau R_T\right),$$

$$R = -\frac{\omega^2}{\sigma} + i\omega + \frac{R_S}{\tau} - \tau R_T = \frac{P}{i\omega},$$

$$S = -\frac{2\omega^2}{\sigma} + 3i\omega + 4(1+\tau) - \left(\frac{R_S}{\tau} - \tau R_T\right) - \frac{4i\tau}{\omega},$$

Substituting these in (3.16) we find

$$\left(\frac{i\omega}{\sigma} + \mu^2\right)\mu^2 A + k(R_T B - R_S C) = \frac{m^2\hat{W}^2}{\sigma P}(kB(i\omega+\tau)\,R_T$$
$$- kR_S C(i\omega+1) - (k^2 S - \mu^2 R)\,A), \tag{4.2a}$$

$$(i\omega+\mu^2)B-Ak = -\frac{m^2\hat{W}^2}{2P}\left(CR_S(i\omega+2)+B\left(-\frac{\omega^2}{\sigma}+i\omega(1-R_T)-2\tau R_T\right)\right.$$
$$\left.+\frac{Ak}{(i\omega+1)}\left(R+S(i\omega+2)\right)\right) \quad (4.2b)$$

$$(i\omega+\tau\mu^2)C-Ak = -\frac{m^2\hat{W}^2}{2P}\left(-\frac{BR_T}{\tau}(i\omega+2\tau)+C\left(-\frac{\omega^2}{\sigma}+i\omega\left(1+\frac{R_S}{\tau}\right)+\frac{2R_S}{\tau}\right)\right.$$
$$\left.+\frac{Ak}{(i\omega+\tau)}\left(R+\frac{S}{\tau}(i\omega+2\tau)\right)\right). \quad (4.2c)$$

Making use of the fact that μ is small, we can easily solve equation (4.2) to $O(\mu^2)$ and we find

$$\omega^2 = \sigma\frac{k^2}{\mu^2}(R_T-R_S)+G\mu^2, \tag{4.3}$$

where G is a complex number which will be found when we solve (4.2) to $O(\mu^4)$. The condition for instability will be that $\mathscr{I}(G) < 0$. We also find from (4.2) that

$$B = \frac{kA}{i\omega}+O(\mu^2) \quad C = \frac{kA}{i\omega}+O(\mu^2). \tag{4.4}$$

Then equations (3.14) and (3.15) give

$$\left.\begin{aligned} A_1 &= \frac{m\hat{W}A}{2\omega}+O(\mu^2), \quad D_1 = \frac{m\hat{W}A}{2\omega}+O(\mu^2),\\ B_1 &= -\frac{m\hat{W}A}{2\omega}+O(\mu^2), \quad E_1 = -\frac{m\hat{W}A}{2\omega}+O(\mu^2),\\ C_1 &= -\frac{m\hat{W}A}{2\omega}+O(\mu^2), \quad F_1 = -\frac{m\hat{W}A}{2\omega}+O(\mu^2) \end{aligned}\right\} \tag{4.5}$$

We are now in a position to check the validity of the approximation which led to equation (3.11). For equation (3.2a) we find, omitting the exponential factor $e^{imz+i\omega t}$

$$Mu_m = -\frac{\hat{W}}{2\sigma}\mathscr{R}(mA(e^{i(k+1)x}-e^{i(k-1)x}))$$

and

$$Mu_r-\langle Mu_r\rangle = -\frac{\hat{W}}{2\sigma}\mathscr{R}\left(\frac{im^2k\hat{W}A}{\omega}(e^{i(k+2)x}-e^{i(k-2)x})\right).$$

Thus, provided $\mu^2 = (k^2+m^2) \ll 1$, then

$$(Mu_r-\langle Mu_r\rangle) \ll Mu_m.$$

For equation (3.2b)

$$Mu_m = -\frac{\hat{W}}{2}\mathscr{R}\left(-\frac{mkA}{\omega}(e^{i(k+1)x}-e^{i(k-1)x})+imA(e^{i(k+1)x}+e^{i(k-1)x})\right)$$

and

$$Mu_r-\langle Mu_r\rangle = -\frac{\hat{W}}{2}\mathscr{R}\left(\frac{m^2\hat{W}A}{2\omega}O(\mu^2)\right).$$

Thus, for (3.2*b*), the temperature equation,

$$(Mu_r - \langle Mu_r \rangle) \ll Mu_m$$

if $\mu^2 \ll 1$. Similarly, the approximation is valid for equation (3.2*c*), the salt equation. We now assume $k^2/\mu^2 = \sin^2\theta = O(1)$, as well as $\sigma \gg 1$, so

$$\sigma \frac{k^2}{\mu^2}(R_T - R_S) \gg 1.\dagger$$

Then, we find to $O(\mu^4)$ and to highest order in σ, that (4.2) has a solution only if

$$i\omega\sigma^2 \sin^2\theta(R_T - R_S) + \sigma^2 \sin^2\theta(R_T - R_S)\, q + i\omega G q - \sigma G \sin^2\theta(R_T - R_S) \\ + \tfrac{1}{2}\hat{W}^2 \cos^2\theta\{-3i\omega\sigma \sin^2\theta + \sigma \sin^2\theta . p\} = 0, \quad (4.6)$$

where

$$q = 1 + \tau - \sin^2\theta(R_T - R_S) + \frac{R_S}{\tau} - \tau R_T,$$

$$p = 7(R_T - R_S) + 2\left(\frac{R_S}{\tau} - \tau R_T\right) + 2\sin^2\theta(R_T - R_S) - 4(1+\tau).$$

If we write $G = a + ib$, equating real and imaginary parts in (4.6), we find

$$\omega b = \sigma \sin^2\theta[\sigma(R_T - R_S) - \tfrac{3}{2}\hat{W}^2 \cos^2\theta].$$

If $b < 0$ then the system is unstable, so the system is unstable if

$$\frac{\hat{W}^2 \cos^2\theta}{2\sigma(R_T - R_S)} > \frac{1}{3}. \quad (4.7)$$

Now the direction in which the wave is travelling is arbitrary. Thus, we see the system will first become unstable when $\cos^2\theta \to 1$, i.e. $\theta \to 0$. So the system is most unstable to waves with a vertical wavenumber vector, which, thus, transport energy horizontally. The instability criterion (4.7) can be re-expressed in dimensional quantities. Using (2.8*a*), (2.10) and (2.11), we find

$$\hat{W}^2 = \frac{2gl^4}{\nu K_T^2}(\beta F_S - \alpha F_T).$$

Then, we find, rearranging (4.7), the system is unstable if

$$\frac{\beta F_S - \alpha F_T}{\nu(\alpha T_z - \beta S_z)} > \frac{1}{3}. \quad (4.8)$$

Having obtained this result, it is possible to trace the problem backwards in order to determine which terms affect the final result. When we do this, we find that the Reynolds-stress term in the momentum equation (4.2*a*) is negligible, which is a consequence of assuming the Prandtl number to be large. However, in the heat and salt

† It is not necessary to make this assumption. We find later that the most unstable wave occurs for $k^2/\mu^2 \to 0$. Thus with R_T and R_S both $O(1)$ and $\sigma \gg 1$, $\sigma k^2/\mu^2$ may vary between zero and infinity. The effect of not making this assumption has been studied. However, the result of the study was that the most unstable wave does occur for

$$\sigma \frac{k^2}{\mu^2}(R_T - R_S) \gg 1.$$

equations, we find that the two forcing terms $-\hat{W}\sin x\,\partial T/\partial z$ and $\hat{T}\cos x\,\partial\psi/\partial z$ and the equivalent terms from the salt equation are both important in determining the stability. Consequently, it is not correct to assume that the flux remains constant in the salt finger. We can calculate the change in the fluxes due to the perturbation. The overall heat flux is given by

$$\mathbf{F}_T = K_T T_z \overline{\left(-\frac{\partial\psi}{\partial x}T\right)}\,\mathbf{k} + K_T T_z \overline{\left(\frac{\partial\psi}{\partial z}T\right)}\,\mathbf{j},$$

where $\mathbf{k}$ is a vertical unit vector and $\mathbf{j}$ is a horizontal unit vector. Then we find

$$\mathbf{F}_T = K_T T_z \frac{\mathbf{k}}{2}\mathscr{R}(\hat{W}^2 + kBA + A_1B_1(k-1) - D_1E_1(k+1)) + K_T T_z \frac{\mathbf{j}}{2}\,\mathscr{R}(-mAB - A_1B_1m + D_1E_1m). \quad (4.9)$$

We shall assume the magnitude of the velocity perturbation of the internal wave, A, is known and is real. The frequency ω is real to zeroth order in μ. Thus, using the relationships (4.4) and (4.5), we find

$$\mathbf{F}_T = -K_T T_z \frac{\hat{W}\hat{T}}{2}\left(1 + \frac{m^2A^2}{2\omega^2} + O(\mu^3)\right)\mathbf{k} + \frac{\mathbf{j}}{2}.O(\mu^3). \quad (4.10)$$

Similarly, the overall salt flux,

$$\mathbf{F}_S = -K_T S_z \frac{\hat{W}\hat{S}}{2}\left(1 + \frac{m^2A^2}{2\omega^2} + O(\mu^3)\right)\mathbf{k} + \frac{\mathbf{j}}{2}.O(\mu^3) \quad (4.11)$$

Now $-K_T T_z \hat{W}\hat{T}/2$ and $-K_T S_z \hat{W}\hat{S}/2$ are the vertical heat and salt fluxes in the absence of salt fingers, as given in (2.10) and (2.11). Thus the magnitudes of $\mathbf{F}_S$ and $\mathbf{F}_T$ are both increased by the internal wave perturbation, although the value of $|\mathbf{F}_S|/|\mathbf{F}_T|$ remains constant. The fact that the fluxes increase indicates that more potential energy is being lost by the salt field and more potential energy is being gained by the temperature field. So more energy is being released by the salt field in order to drive the motion.

5. Discussion

Many experiments have been performed which display instabilities of salt fingers. So it is of interest to compare the criterion for instability obtained here with some laboratory experiments. The most relevant experiment which has been done is that of Stern & Turner (1969). They use salt and sugar, rather than heat and salt, as the two diffusing substances, because it is experimentally simpler. We shall still, however, refer to them as heat, T, and salt, S. They set up a very deep layer of fresh water with a uniform temperature gradient $T_z > 0$ and surface temperature $\overline{T}(0)$ beneath another deep layer of uniform temperature $\overline{T}(0)+\Delta T$ and salinity ΔS. The density of the upper layer is less than that of the lower, so $\alpha\Delta T > \beta\Delta S$. Salt fingers form at the interface as soon as the two layers are formed, and they penetrate into the lower fluid. If the experiment is repeated with a smaller value of T_z, and the same values of ΔS and ΔT, then initially long fingers form. However, after a short time, the fingers just below the interface become unstable and give way to a well-stirred convective layer, which is maintained by the flux through a thin salt-finger layer at the interface. If the temperature gradient, T_z, is reduced further, a second layer can become unstable beneath the first layer. By suitable choices of the parameters ΔT, ΔS and T_z it is possible to obtain several convecting layers, each bounded above and below by a relatively thin layer of salt fingers. Layers like these have been observed in the ocean by several authors (e.g. Tait & Howe 1968, 1971).

In order to compare this experiment with the theoretical stability criterion, it is necessary to know the salt flux, F_S, through the fingers. There is a fairly well-documented relationship between F_S and ΔS (Turner 1967)

$$F_S = C(\beta\Delta S)^{\frac{4}{3}}. \tag{5.1}$$

The number C in this relationship may vary with K_S/K_T and $\beta\Delta S/\alpha\Delta T$ and is determined experimentally. Applying the relationship (5.1) to Stern & Turner's (1969) experiment using their value of $C = 10^{-2}$ cm s^{-1}, we find that if

$$(\beta F_S - \alpha F_T)/\nu(\alpha T_z - \beta S_z) \gtrsim 2{\cdot}8,$$

the system is unstable and that if $(\beta F_S - \alpha F_T)/\nu(\alpha T_z - \beta S_z) \lesssim 1{\cdot}2$, the system is stable. The experiments show that the parameter grouping given is that which determines the system's stability. This indicates that it is the collective instability mechanism that is leading to the break up of the fingers.

Other experiments have been carried out on the thin salt-finger layer that exists between two convecting regions. The system is set up by placing a very deep uniform temperature $T_0 - \frac{1}{2}\Delta T$ and salinity $S_0 - \frac{1}{2}\Delta S$ beneath another deep layer of temperature $T_0 + \frac{1}{2}\Delta T$ and salinity $S_0 + \frac{1}{2}\Delta S$. The unstable salt fingers drive the convecting regions and consequently one expects the salt finger interface to be marginally stable. Schmitt (1979) and Linden (1973) performed this experiment using heat and sugar and they found values of $(\beta F_S - \alpha F_T)/\nu(\alpha T_z - \beta S_z)$ ranging from 0·2–1·9 in the salt-finger layer. Lambert & Demenkow (1972) performed the same experiment but using salt and sugar. They found $(\beta F_S - \alpha F_T)/\nu(\alpha T_z - \beta S_z)$ was approximately 2×10^{-3} in their experiments. This same experiment has been carried out more carefully recently by Griffiths & Ruddick (1980), and it shows that C decreases as $\Delta T/\Delta S$ increases, and their results lead to an increase in the above factor by an order of magnitude.

So we see from the experiments that have been made on salt fingers that there is insufficient evidence to decide whether the theoretical stability criterion is in agreement with experiment, mainly because the theoretical model is not close enough to the experiments. Nevertheless, it can be seen from the experiments that it is the value of $(\beta F_S - \alpha F_T)/\nu(\alpha T_z - \beta S_z)$ which determines the stability of the system. Unfortunately, none of the experiments yet performed allow one to examine the onset of the instability, and, although the estimates for the parameter C are improving, the stability criterion cannot really be checked until the heat and salt fluxes can be found more accurately.

6. Conclusions

We have shown that a system of two-dimensional salt fingers is unstable to internal wave perturbations of long wavelength provided

$$\frac{\beta F_S - \alpha F_T}{\nu(\alpha T_z - \beta S_z)} > \frac{1}{3} \tag{6.1}$$

where F_T and F_S are the heat and salt fluxes of the salt fingers, ν is the kinematic viscosity of the fluid and T_z and S_z are the heat and salt gradients in the fluid. The internal wave which first makes the system unstable is found to be propagating almost horizontally. The ordering of the parameters in the problem has been chosen so that the Prandtl number, σ, is large, the ratio of the two diffusivities $\tau = K_S/K_T$, is $O(1)$ or less, and the wavenumber, μ, of the internal wave perturbation is small. Although the most unstable wave occurs when $k/\mu \to 0$, it also has $\sigma k^2/\mu^2 \gg 1$, i.e. $k^2/\mu^2 \gg 1/\sigma^2$, so the wave is propagating close to horizontally, on a scale determined by σ. The internal wave perturbation has been shown to increase the heat and salt fluxes through the system so more energy is released by the unstable salt field and more is gained by the temperature field.

Unfortunately, the experiments which have been performed on the instabilities of salt fingers have not been sufficiently accurate to determine whether the stability criterion (6.1) is in agreement with experiment. The main restriction of the model used here is that only two-dimensional motions are allowed. If three-dimensional salt fingers of square planform were studied, it is possible that the number on the right-hand side of (6.1) would be altered.

Another problem which could be usefully tackled would be a numerical approach to this problem, so that we could study not only long wave perturbations, but perturbations of any wavelength. This should be possible by using a method similar to that of Roberts (1972). From such a study one should find the wavelength of perturbation which gives the maximum growth rate. One would expect this wavelength to be related to the distance between layers in salt-finger experiments.

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